Free convection of fluid in a vertical tube filled with porous material

By R. A. WOODING
Applied Mathematics Laboratory, D.S.I.R., Wellington, N.Z.

(Received 18 September 1961 and in revised form 19 January 1962)

The problem of an unstable fluid overturning in a vertical tube filled with porous material is treated by an approximation of boundary-layer type. It is shown that the fluid can experience a pseudo-inertial effect, in which variations in density across the tube exhibit properties analogous to variations of momentum in an inertial flow. The mean fluid density and mean-square vertical velocity over a horizontal cross-section of the tube are related by a pair of hyperbolic equations, for which there exist two systems of characteristics. It is shown that changes in the mean density of the fluid can be propagated as discontinuities. For discontinuities of finite amplitude, two jump conditions are derived, one of which is found to involve an undetermined parameter λ. The theory is applied to the case of a vertical tube containing porous material saturated with water, which is attached at the top to a reservoir filled with an aqueous solution (an analogue of Taylor's (1954) experiment). The motion of a finite discontinuity which arises at the initial unstable interface is calculated by two approximate methods. These results compare satisfactorily with the data from three experiments, using tubes of circular cross-section, provided that the value of λ is about 0·75. If the theoretical interpretation is correct, it appears that convective flow ceases when the vertical density gradient is slightly less than the neutral value.

1. Introduction

In experimental work on Rayleigh instability of a fluid in a vertical tube, Sir Geoffrey Taylor (1954) employed a capillary tube filled with water and closed at the bottom, and connected at the top to a reservoir containing an aqueous solution of greater density than that of water. Convection currents developed in the tube from disturbances to the unstable interface between the two fluids. If the denser fluid in the upper reservoir was marked with a suitable dye, its motion could be seen to take the form of a long column which descended in the tube and displaced the lower fluid. It appeared that there existed a finite discontinuity, or jump, in the average fluid properties at the leading edge of the column. When the density gradient no longer exceeded the critical value at any point, the convective flow ceased. The vertical density gradient of the fluid in the tube could then be measured.

Some use has also been made of this experimental method for the determination of the density gradient at neutral stability of a fluid in a vertical tube filled with porous material and in a vertical Hele–Shaw cell, where similar experimental phenomena have been observed (Wooding 1959, 1960).

Now, the possibility exists that the convective motion could lead to a final equilibrium situation in which the vertical density gradient has a value less than the critical value; i.e. any one of the experiments mentioned above should be regarded as giving a lower bound for the true neutral gradient. A difference between the experimental result and the neutral value could arise from two causes. The first is molecular diffusion along the tube after convection has practically ceased. This effect is small, leading to an error of the order of the tube diameter in the measurement of the total depth of descent of the overlying fluid into the lower, less dense, fluid. The second possible phenomenon, suggested privately by Mr C. H. Bosanquet, is that the denser fluid 'overshoots' during the period of convective flow. The existence of such a phenomenon might indicate that the fluid is more unstable for finite disturbances than for infinitesimal disturbances, and leads to the suggestion, put forward by Taylor and Bosanquet, that the effect might be eliminated by increasing the density of the fluid in the upper reservoir very gradually to the desired value.

Some knowledge of the convective processes which take place before equilibrium is reached in the tube would be useful; and a simple approximate theory is described here for the case of free convection in a fluid of variable density in which molecular diffusion is important when the fluid is contained in a vertical tube filled with porous material and closed at the bottom. In the development of the theory, the Péclet number of the flow is taken to be $O(1)$ or less, and the flow is assumed to possess the following properties. I. The characteristic length scale of the motion along the tube is much greater then the length scale in any direction normal to the tube axis. II. The motion has approached a 'quasi-steady' state in which the mean vertical density gradient departs only slightly from the critical value. III. The flow pattern resembles a column-like convection which is characteristic of a single disturbance mode, usually the lowest mode, in the theory of Rayleigh instability in a vertical tube. (When the mean vertical density gradient has the critical value, a flow of this type may be present with finite amplitude, and constitutes a simple similarity solution of the equations of motion.) IV. When the 'overall' vertical density gradient exceeds the critical value by a finite amount, one or more finite discontinuities in density and other mean fluid properties will appear. In the neighbourhood of these discontinuities, the assumptions I, II and III break down.

2. The approximate equations of motion

Consider a homogeneous isotropic porous medium contained in a long vertical tube, and saturated with an incompressible fluid of variable density. Rectangular Cartesian co-ordinates $OX_i\,(i = 1, 2, 3)$ are taken with OX_3 directed vertically upwards. Let U_i be the flow velocity component in the direction X_i, P be the pressure, t the time, μ and ρ the viscosity and density of the fluid, κ the effective diffusivity, k and ϵ the permeability and porosity of the medium, and g the acceleration due to gravity. The quantities μ, k and ϵ will be assumed constant, and the convective motion will be assumed to be so slow that the diffusivity κ is isotropic and constant, i.e. due to molecular motion alone. Then it is convenient to introduce the following dimensionless variables, taking b as a typical horizontal dimension of the tube.

$$(x,y,z) = (X_1/b, X_2/b, X_3/b), \quad \tau = \kappa t/\epsilon b^2,$$
$$(u,v,w) = (U_1 b/\kappa, U_2 b/\kappa, U_3 b/\kappa), \quad p = kp/\kappa\mu,$$
$$\eta = \frac{gkb\bar{\rho}}{\kappa\mu}, \quad \vartheta = \frac{gkb(\rho-\bar{\rho})}{\kappa\mu},$$

where

$$\bar{\rho}(z,\tau) = \frac{1}{S}\iint_S \rho \, \mathrm{d}S,$$

the integral being taken over the total area S of a horizontal cross-section of the tube at any given value of z. In terms of these variables, the equations of continuity, motion (Darcy's law) and mass transport are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\left.\begin{aligned} \frac{\partial p}{\partial x}+u &= 0,\\ \frac{\partial p}{\partial y}+v &= 0,\\ \frac{\partial p}{\partial z}+w+\eta+\vartheta &= 0,\end{aligned}\right\} \tag{2}$$

$$\left(\frac{\partial}{\partial \tau}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)(\eta+\vartheta) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\eta+\vartheta). \tag{3}$$

At the impermeable walls of the tube, the relevant boundary conditions are

$$\frac{\partial \vartheta}{\partial n} = 0 \quad \text{and} \quad q_n = 0, \tag{4}$$

where $\partial/\partial n$ signifies the normal derivative and q_n the component of the flow velocity normal to the wall. Since it is assumed that the vertical tube is closed at the bottom, the continuity restriction

$$\bar{w} = 0 \tag{5}$$

must hold at all values of z.

When the dimensionless pressure p is eliminated between the first two equations in (2), a relationship between the velocity components u and v is obtained

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}; \tag{6}$$

i.e. under the action of gravitational forces alone, the z-component of 'vorticity' is zero.

Approximations of boundary-layer type

To render this system amenable to an approximate treatment, it will now be assumed that the Péclet number of the vertical flow (based upon the horizontal length scale b) is of order unity, and the assumptions I, II and III of § 1 will be applied. A flow region which is free of discontinuities is considered.

For three-dimensional flows, the solution for w (for example) can be written in the form of the double series

$$w = \sum_i \sum_j w_{ij}(z, \tau)\, \phi_{ij}(x, y), \tag{7}$$

where ϕ_{ij} is an eigenfunction satisfying the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \alpha_{ij}^2\right) \phi_{ij} = 0 \tag{8}$$

with the boundary condition $\partial \phi_{ij}/\partial x = 0$ at the walls of the tube. In (8), the square of the wave-number α_{ij}^2 is equal to the critical value of the dimensionless density gradient, or Rayleigh number $\partial \eta/\partial z$, at which a small disturbance of form ϕ_{ij} is neutrally stable according to linear theory.

Let the dimensionless quantity s denote the ratio of the characteristic horizontal dimension b of the tube to the length scale of the motion in the direction Oz. From assumption I, it follows that $s \ll 1$. Now, by assumption III, one term of the double series in (7) is predominant; without significant loss of generality, this may be taken to be that term (or combination of terms) for which α_{ij} has its minimum value, α say, a quantity which is not less than $O(1)$ in magnitude. For simplicity, suppose that only one term is involved. Then the orders of magnitude of the terms in equations (1), (2) and (3) may be discussed with respect to the magnitude of this predominant term of (7). The process is justifiable *a posteriori*.

With $w = O(1)$, $z = O(s^{-1})$ and x and $y = O(\alpha^{-1})$, the three terms in the equation of continuity (1) are of comparable magnitude, $O(s)$, if u and v are $O(s/\alpha)$. Then $\partial p/\partial x$ and $\partial p/\partial y$ are $O(s/\alpha)$ from the first two equations in (2). That is, $\partial p/\partial z$ is constant to within $O(s^2/\alpha^2)$ over any horizontal cross-section of the tube. From the third equation in (2) and the condition (5),

$$\frac{\partial p}{\partial z} + \eta = O(s^2/\alpha^2),$$

and

$$w + \vartheta = O(s^2/\alpha^2). \tag{9}$$

In equation (3), the ratio of $\partial^2 \vartheta/\partial z^2$ to $(\partial^2/\partial x^2 + \partial^2/\partial y^2)\,\vartheta$ is $O(s^2/\alpha^2)$. Eliminating ϑ from (3) by means of (9) and dropping terms of $O(s^2/\alpha^2)$ gives the equation

$$\underset{\text{(i)}}{\frac{\partial w}{\partial \tau}} + \underset{\text{(ii)}}{\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) w} = \underset{\text{(iii)}}{\left(\frac{\partial \eta}{\partial z} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w} + \underset{\text{(iv)}}{\frac{\partial \eta}{\partial \tau}} - \underset{\text{(v)}}{\frac{\partial^2 \eta}{\partial z^2}}, \tag{10}$$

where, for convenience, the terms are allocated to groups numbered from (i) to (v). If magnitudes are assigned to the various quantities as before, each of the quadratic terms in (ii) is readily shown to be $O(s)$. By the 'quasi-steady' assumption II, large rates of change are excluded; consequently, it is assumed that the terms (i) and (iv) are of the same order of magnitude as (ii). Also, by II and III, the group of terms in (iii) is taken to be $O(s)$ overall, i.e. for the predominant mode the tendency of the fluid to overturn is almost balanced by the damping due to transverse diffusion. If use is made of (8), this gives

$$\frac{1}{\alpha^2}\left(\frac{\partial\eta}{\partial z}-\alpha^2\right) = O\left(\frac{s}{\alpha^2}\right),$$

and it follows that

$$\frac{1}{\alpha^2}\frac{\partial^2\eta}{\partial z^2} = O\left(\frac{s^2}{\alpha^2}\right).$$

That is, the effect of longitudinal diffusion is small and the term (v) in equation (10) can be neglected.

Finally, it is necessary to examine the interaction of higher-order modes upon the assumed predominant mode. For simplicity, let only two terms in (7) be considered—the predominant mode of $O(1)$ and one higher mode of amplitude w_{ij}. Then the largest possible contribution due to this higher mode, to the quadratic terms (ii) in equation (10), is $O(sw_{ij}\alpha_{ij}/\alpha)$. However, the contribution due to the diffusion terms in (iii) for the higher mode is $O(\alpha_{ij}^2 w_{ij})$ and is negative, so that this mode tends to be heavily damped. In fact, $w_{ij} = O(s/\alpha\alpha_{ij})$. The contribution to the quadratic terms is therefore $O(s^2/\alpha^2)$ and may be neglected. Thus the assumption III is justified, and the flow system closely resembles the predominant mode. This leads to the new equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\alpha^2\right)w = 0 \tag{11}$$

for the vertical velocity w.

The equations (1), (6), (10) and (11) govern the behaviour of the four unknowns u, v, w and η. From (4) and (9), the appropriate boundary conditions are

$$\partial w/\partial n = q_n = 0 \tag{12}$$

at the walls of the tube.

Although the above analysis follows the Prandtl method of boundary-layer approximation, a slight difference in the properties of the system will have been observed. In the conventional boundary-layer case, the system of equations remains determinate by virtue of the condition $\partial p/\partial n = 0$, the pressure p being specified by its known value at the outer edge of the layer. In the present case, the distribution of p is a function of η, one of the unknowns, and it has been necessary to assume a particular form for the flow distribution (cf. equation (11)) in order to render the system determinate.

From the left-hand side of equation (10), one observes that the fluid can experience a 'pseudo-inertial' effect, with variations in density across the tube exhibiting properties analogous to variations of momentum in an inertial flow. If W (in dimensional units) is a typical velocity, the Péclet number $w = Wb/\kappa$ is analogous to the Reynolds number of the corresponding momentum flow.

Integrated form of the approximate equations

Two integral relationships can be derived from equation (10). If (10) is integrated with respect to x and y over the tube cross-section, using (1), (5) and the boundary conditions (12), one obtains, after application of Green's theorem,

$$\frac{\partial\eta}{\partial\tau} = \frac{\partial}{\partial z}(\overline{w^2}). \tag{13}$$

Similarly, if (10) is first multiplied by w and then integrated with respect to x and y, there results

$$\frac{1}{2}\frac{\partial}{\partial \tau}(\overline{w^2})+\frac{1}{2}\frac{\partial}{\partial z}(\overline{w^3}) = -\overline{\left(\frac{\partial w}{\partial x}\right)^2}-\overline{\left(\frac{\partial w}{\partial y}\right)^2}+\overline{w^2}\frac{\partial \eta}{\partial z}. \tag{14}$$

From (11), the first two terms on the right-hand side of this equation can be replaced by $-\alpha^2\overline{w^2}$. Also, it is convenient to introduce the further assumption that the tube cross-section is of a simple symmetrical form; in that case the fluid motion is almost purely antisymmetric if the most unstable mode predominates, and the term involving $\overline{w^3}$ in (14) vanishes. Equation (14) becomes

$$\frac{1}{2}\frac{\partial}{\partial \tau}(\overline{w^2}) = \overline{w^2}\left(\frac{\partial \eta}{\partial z}-\alpha^2\right). \tag{15}$$

Equations (13) and (15) form a hyperbolic system, and can be written in the integrated characteristic form

$$dz/d\tau = \pm(2\overline{w^2})^{\frac{1}{2}}, \quad \eta-\alpha^2 z\mp(2\overline{w^2})^{\frac{1}{2}} = \text{constant}. \tag{16}$$

In (16), it is convenient to adopt the convention that the square root is positive, and to take either the two upper signs together, or the two lower signs together, in defining each of the two sets of characteristics.

Properties of the approximate equations

The equations (13) and (15), or equations (16), possess the following similarity property. Suppose that $\Delta\eta$ is the measure of a typical density difference. Then, if the amplitudes of the quantities, η, $(\overline{w^2})^{\frac{1}{2}}$ and z are scaled in proportion to $\Delta\eta$, it is found that the scaled equations remain invariant.

In equations (16), let

$$\eta-\alpha^2 z+(2\overline{w^2})^{\frac{1}{2}} = r,$$

and

$$\eta-\alpha^2 z-(2\overline{w^2})^{\frac{1}{2}} = -s,$$

so that $(2\overline{w^2})^{\frac{1}{2}} = \frac{1}{2}(r+s)$ and $\eta-\alpha^2 z = \frac{1}{2}(r-s)$. The quantities r and s are Riemann invariants (Courant & Friedrichs 1948, §37). Then

$$\left.\begin{aligned}\frac{\partial z}{\partial r} &= \tfrac{1}{2}(r+s)\frac{\partial \tau}{\partial r},\\ \frac{\partial z}{\partial s} &= -\tfrac{1}{2}(r+s)\frac{\partial \tau}{\partial s},\end{aligned}\right\} \tag{17}$$

and z can be eliminated between these last two equations to give

$$2\frac{\partial^2 \tau}{\partial r\,\partial s}+\frac{1}{r+s}\left(\frac{\partial \tau}{\partial r}+\frac{\partial \tau}{\partial s}\right) = 0. \tag{18}$$

If (18) is solved for $\tau(r,s)$, the solution for $z(r,s)$ follows from (17). Equation (18) is identical with the Riemann equation for $\tau(r,s)$ in the case of one-dimensional isentropic flow of a gas with adiabatic constant $\gamma = 2$. Although the same equivalence does not hold for the z-equation (since the slope of each of the two characteristics through a point differs, in the present problem, from the slope in the gas flow case by a quantity $r-s$), the qualitative properties of the two flows can be expected to be very similar.

This may be compared with the case of propagation of shallow-water waves (Stoker 1948) where the governing equations are completely equivalent to the equations of the same one-dimensional isentropic gas flow.

3. Propagation of density discontinuities

In § 1, it was noted that discontinuities in the mean properties (density and velocity) of the convecting fluid had been observed experimentally. The possibility of such discontinuities appearing can be demonstrated from the close resemblance of the approximate equations of the flow to the gas flow equations. For example, using a method of Courant & Friedrichs (1948), it can be shown that a wave transporting an increase in mean velocity will tend to steepen at the front in the manner of a compression wave in a gas, while the opposite type of wave will tend to flatten out, as in a rarefaction wave. Thus a theory analogous to the theory of one-dimensional isentropic gas flow can be derived, in which discontinuities analogous to shock waves are permitted.

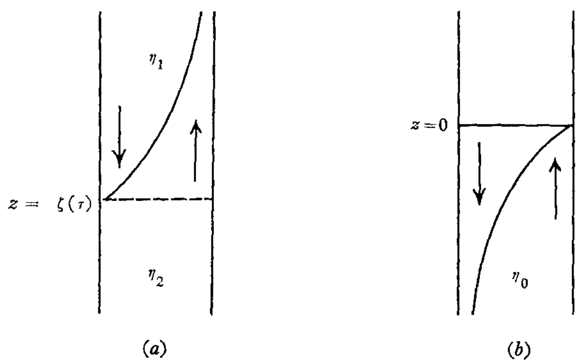

FIGURE 1. (*a*) Descending discontinuity in a vertical tube filled with porous material. (*b*) Stationary discontinuity at the top of the porous medium in a vertical tube, with overlying fluid of constant density.

Jump conditions

The motion of a discontinuity of finite amplitude will now be considered. This discontinuity must be of finite thickness, as in the case of a bore in shallow water (Stoker 1948) or a kinematic shock (Lighthill & Whitham 1955), since a transition region exists in which the approximate theory breaks down. In the present treatment, the thickness of this transition region will be neglected.

Suppose that a discontinuity in fluid properties is located at $z = \zeta(\tau)$ (in figure 1(*a*)) and that the mean density and the flow velocity are denoted by η_1 and w_1 above the jump and by η_2 and w_2 below the jump. It is convenient to refer the system to a coordinate frame (z', τ) moving with the jump, i.e. $z = \zeta(\tau)+z'$. Then, relative to this frame, the flow process in the neighbourhood of the jump can be considered steady, giving

$$\frac{\partial}{\partial \tau} = -\dot{\zeta}\frac{\partial}{\partial z'} \quad (\dot{\zeta} \equiv d\zeta/d\tau)$$

so that the continuity law (13) becomes

$$-\dot{\zeta}\frac{\partial \eta}{\partial z'} = \frac{\partial \overline{w^2}}{\partial z'}.$$

When this equation has been integrated across the jump from side 1 to side 2, there results

$$\dot{\zeta} = -\frac{\overline{w_1^2} - \overline{w_2^2}}{\eta_1 - \eta_2}. \tag{19}$$

Equation (15) can be treated similarly; thus

$$\tfrac{1}{2}\dot{\zeta}\frac{\partial \overline{w^2}}{\partial z'} = \overline{w^2}\left(\frac{1}{\dot{\zeta}}\frac{\partial \overline{w^2}}{\partial z'} + \alpha^2\right).$$

A unique jump condition is, however, not obtained; for if this expression is multiplied by $(\overline{w^2})^{k-1}$ and integrated across the jump, the relation

$$\dot{\zeta}^2 = \frac{2k}{k+1}\frac{(\overline{w_1^2})^{k+1} - (\overline{w_2^2})^{k+1}}{(\overline{w_1^2})^k - (\overline{w_2^2})^k} \tag{20}$$

is found for arbitrary positive k.

When the discontinuity becomes very weak, $\overline{w_2^2} \to \overline{w_1^2}$ and (20) gives $\dot{\zeta}^2 = 2\overline{w_1^2}$, which corresponds to the characteristic velocity.

For the motion of a jump descending into still fluid, let $\overline{w_2^2} \to 0$. Then

$$\dot{\zeta}^2 = \lambda \overline{w_1^2}, \tag{21}$$

where the parameter $\lambda = 2k/(k+1)$. (Thus $0 < \lambda < 2$.) A comparison of (19) and (21) for this case shows that

$$\overline{w_1^2} = \lambda(\eta_1 - \eta_2)^2. \tag{22}$$

From the relations (19) to (22), it is evident that the velocities associated with the jump are scaled in proportion to $\eta_1 - \eta_2$, in accordance with the similarity law obeyed by equations (13) and (15). (A similar variation of the length scale of the motion near the discontinuity has been observed experimentally.)

It is also important to consider the boundary conditions which apply at the top of a porous medium in a long tube (at $z = 0$ in figure 1(*b*)). The overlying fluid is assumed to be of constant density η^*, while the condition of uniform pressure gives $\partial w/\partial z = 0$ at $z = 0$. However, since the convective motion in the tube involves ascending fluid of density less than η^*, it follows that the mean density η_0, evaluated just below $z = 0$, is less than η^*. Hence some kind of jump, stationary in position but perhaps not stationary in length scale, must be assumed to exist at $z = 0$.

If a 'quasi-steady' condition is assumed to hold in the neighbourhood of $z = 0$, the density difference $\eta^* - \eta_0$ can be taken constant over a short period of time. Equation (13) gives $\partial \overline{w^2}/\partial z = 0$ at small negative z (putting $\partial/\partial\tau = 0$) while (15) leads back to the condition that the density gradient is very nearly critical. Then, if the similarity property is assumed to hold,

$$\frac{\overline{w_0^2}}{(\eta^* - \eta_0)^2} = \lambda', \tag{23}$$

where w_0 is the velocity at $z = 0$ and λ' is a constant. This expression is of the same form as (22).

The appearance of undetermined quantities λ (or k) and λ' in the above formulae introduces a difficulty which has not been overcome, since the detailed structure of the transition regions is unknown. It is also probable that the numerical values of these quantities are not the same for tubes of different cross-sectional shapes. In the remainder of this paper, it will be assumed that

$$\lambda' = \lambda, \tag{24}$$

and the value of λ will be adjusted to give optimum agreement with experimental data. The assumption (24) is at least partially justified by the success of this approach.

4. Discontinuity descending into still fluid

Suppose that a vertical tube is filled, in the region for which $z \leqslant 0$, with a porous medium saturated with a static liquid of dimensionless density $\eta = \beta z + \eta_c$, where η_c is a constant density and where $\beta < \alpha^2$ for stability, while the region for which $z > 0$ contains overlying fluid of constant density $\eta^* > \eta_c$. Then the motion of a descending discontinuity which leaves the origin $z = 0$ at time $\tau = 0$ is described by the characteristic equations (16), the jump conditions (19) and (21), and the 'boundary jump condition' (23) with $\lambda' = \lambda$. The following scaled variables are now introduced.

$$\left.\begin{aligned} R &= \frac{\eta - \beta z - \eta_c}{\eta^* - \eta_c}, \quad W = \frac{(2\overline{w^2})^{\frac{1}{2}}}{\eta^* - \eta_c}, \quad T = (\alpha^2 - \beta)\,\tau, \\ Z &= \frac{(\alpha^2 - \beta)\,z}{\eta^* - \eta_c}, \quad Z_1 = \frac{(\alpha^2 - \beta)\,\zeta}{\eta^* - \eta_c}. \end{aligned}\right\} \tag{25}$$

Here Z_1 indicates the space co-ordinate of the jump. It is also convenient to designate the time T as T_1 at the jump; then, in the (Z, T)-plane a given characteristic will intersect the jump at the point (Z_1, T_1) and the line $Z = 0$ at time T_0.

The equations (16) become

$$dZ/dT = -W, \quad R - Z + W = R_0 + W_0, \tag{26}$$

$$dZ/dT = +W, \quad R - Z - W = R_0 - W_0, \tag{27}$$

upon descending and ascending characteristics respectively, where R_0 and W_0 are the values of R and W in the porous medium just below $Z = 0$. The jump conditions (19) and (21) give

$$\frac{dZ_1}{dT_1} = -\frac{W_1^2}{2R_1} = -\lambda R_1 \tag{28}$$

since $R_2 = W_2 = 0$ ahead of the jump, while the boundary jump condition (23) at $Z = 0$ becomes

$$W_0 = (2\lambda)^{\frac{1}{2}}\,(1 - R_0). \tag{29}$$

When $T_1 = 0$, the quantities R_1 and W_1 behind the jump are identical with R_0 and W_0 at $Z = 0$; combining (28) with (29) shows that the initial values at $T_1 = T_0 = 0$ are

$$\left.\begin{aligned} R_0 = R_1 = \tfrac{1}{2}, \quad W_0 = W_1 = \tfrac{1}{2}(2\lambda)^{\frac{1}{2}}; \\ dZ_1/dT_1 = -\tfrac{1}{2}\lambda, \quad Z_1 = 0. \end{aligned}\right\} \tag{30}$$

Two methods of obtaining approximate solutions of the system of equations (26) to (30) will now be considered.

Perturbation method

Suppose that $\delta = 1-(2\lambda)^{-\frac{1}{2}}$ and consider the formal perturbation scheme

$$Z = Z^{(0)} + \delta Z^{(1)} + \ldots, \tag{31}$$

with similar expansions for R and W, which is valid for $\delta \to 0$. If R_0 is eliminated using (29), the equations for the zero-order coefficients are

$$dZ^{(0)}/dT = -W^{(0)}, \quad R^{(0)} - Z^{(0)} + W^{(0)} = 1, \tag{32}$$

$$dZ^{(0)}/dT = W^{(0)}, \quad R^{(0)} - Z^{(0)} - W^{(0)} = 1 - 2W_0^{(0)}; \tag{33}$$

$$dZ_1^{(0)}/dT_1 = -\tfrac{1}{2}R_1^{(0)}, \quad R_1^{(0)} = W_1^{(0)}; \tag{34}$$

at

$$T_0 = T_1 = 0, \quad R_1^{(0)} = W_0^{(0)} = W_1^{(0)} = \tfrac{1}{2}, \quad dZ_1^{(0)}/dT_1 = -\tfrac{1}{4}. \tag{35}$$

For the first-order coefficients, the equations

$$\left.\begin{aligned} R^{(1)} - Z^{(1)} + W^{(1)} = W_0^{(0)} \\ dZ_1^{(1)}/dT_1 = -\tfrac{1}{2}W_1^{(1)} - \tfrac{1}{2}W_1^{(0)}, \quad R_1^{(1)} = W_1^{(1)} - W_1^{(0)}, \\ Z_1^{(1)} = 0 \quad \text{at} \quad T_1 = 0, \end{aligned}\right\} \tag{36}$$

are required in order to determine the behaviour of the function $Z_1^{(1)}(T_1)$.

In the region behind the discontinuity, a comparison of the second equations in (32) and (33) reveals that $W^{(0)} = W_0^{(0)}$ upon an ascending characteristic, i.e. in the zero-order approximation the jump is followed by a simple-wave region in which the ascending characterisics are straight lines. This becomes the exact solution in the case $\lambda = \frac{1}{2}\,(\delta = 0)$. Conversely, it is easily shown that the exact solution cannot be of simple-wave type when $\lambda \neq \frac{1}{2}$.

When (32) and (34) are applied to the flow quantities just behind the jump, there results

$$2R_1^{(0)} = 1 + Z_1^{(0)}, \tag{37}$$

so that the strength of the discontinuity decreases, approximately linearly with depth to zero at $Z_1^{(0)} = -1$. From (34) and (37), a first-order differential equation is obtained for $Z_1^{(0)}$; the solution is

$$Z_1^{(0)} = e^{-\frac{1}{4}T_1} - 1. \tag{38}$$

It may be noted also that, when the discontinuity is advancing into neutrally stable fluid ($\beta \to \alpha^2$), the rate of descent tends to a constant with $Z_1^{(0)} \to -\frac{1}{4}T_1$.

Each ascending characteristic obeys a straight-line equation of the form

$$Z^{(0)} - Z_1^{(0)} = W_1^{(0)}(T - T_1).$$

If the above equations and results are used to eliminate the quantities $Z_1^{(0)}$, $W_1^{(0)}$ and T_1, the relationship between $W^{(0)}$, $Z^{(0)}$ and T in the region $Z_1^{(0)} < Z^{(0)} < 0$ is found to be

$$Z^{(0)}+1 = W^{(0)}(T+2+4\log 2W^{(0)}). \tag{39}$$

For the descending characteristics, (39) and the first equation in (32) are combined. This leads to the parametric equations

$$\left.\begin{aligned} T &= (W_0^{(0)-\frac{1}{2}} - 4W_0^{(0)\frac{1}{2}})\, W^{(0)-\frac{1}{2}} + 2 - 4\log 2W^{(0)}, \\ Z^{(0)}+1 &= (W_0^{(0)-\frac{1}{2}} - 4W_0^{(0)\frac{1}{2}})\, W^{(0)\frac{1}{2}} + 4W^{(0)} \end{aligned}\right\} \tag{40}$$

for T and $Z^{(0)}$ in terms of $W^{(0)}$, where $W_0^{(0)}$ is evaluated at the point of intersection of the given characteristic with the line $Z^{(0)} = 0$.

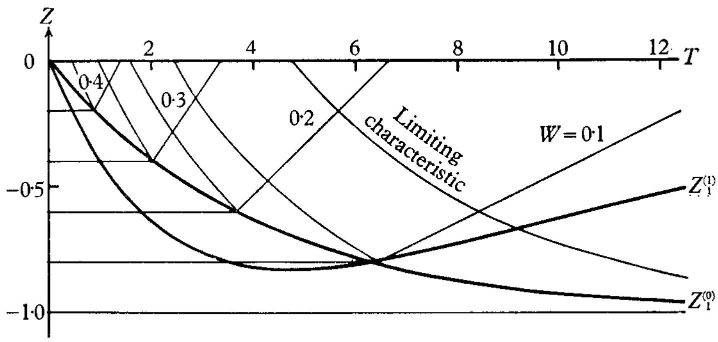

FIGURE 2. Zero- and first-order perturbation coefficients $Z_1^{(0)}$ and $Z_1^{(1)}$ for a discontinuity descending into still fluid.

Figure 2 illustrates graphically the zero-order solution. An interesting feature is that there exists a 'limiting' descending characteristic which intersects the discontinuity only when $T_1 \to \infty$, and which intersects $Z^{(0)} = 0$ when

$$T_0 = 2+4\log 2.$$

Subsequent descending characteristics do not intersect the jump and presumably do not influence its motion.

When (37) is combined with the second equations in (32) and (40), there results

$$2W_1^{(0)\frac{1}{2}} = 4W_0^{(0)\frac{1}{2}} - W_0^{(0)-\frac{1}{2}} \tag{41}$$

upon a descending characteristic. Then, a differential equation for the first-order coefficient $Z_1^{(1)}$ in terms of $W_1^{(0)}$ can be obtained by eliminating $R_1^{(1)}$, $W_1^{(1)}$, $W_0^{(0)}$ and T_1 from (41), (36) and the relation

$$T_1 = -4\log 2W_1^{(0)}. \tag{42}$$

The solution which satisfies the initial condition $Z_1^{(1)} = 0$ is

$$Z_1^{(1)} = -\tfrac{1}{4} + \tfrac{1}{8}W_1^{(0)}\left\{10 + 25\log 2W_1^{(0)} + \log\left[\frac{1}{2}\frac{(1+4/W_1^{(0)})^{\frac{1}{2}}+1}{(1+4/W_1^{(0)})^{\frac{1}{2}}-1}\right] - 2(1+4/W_1^{(0)})^{\frac{1}{2}}\right\}; \tag{43}$$

(42) and (43) are parametric relations for T_1 and $Z_1^{(1)}$ in terms of $W_1^{(0)}$. The function $Z_1^{(1)}(T_1)$ is plotted in figure 2. When $\lambda > \frac{1}{2}(\delta > 0)$, it is found that the rate of descent dZ_1/dT_1 must become zero at a finite value of T_1; the value of Z_1 at that point will be taken as the final depth of descent, and the apparent 'retreat' of the jump at later times will be ignored.

Method of Whitham and Rościszewski

Now consider the original equations (26) to (30), and suppose that the ascending characteristics may be replaced by straight lines. This is the method used by Rościszewski (1960) in extending Whitham's characteristic rule (Whitham 1958). In (27) and (28), let $W = W_1 = W_0$; it follows that

$$2R_1 = 1 + Z_1, \tag{45}$$

which corresponds to (37) in the perturbation scheme. The corresponding approximate solution for the jump motion (cf. (38)) is

$$Z_1 = e^{-\frac{1}{2}\lambda T_1} - 1. \tag{46}$$

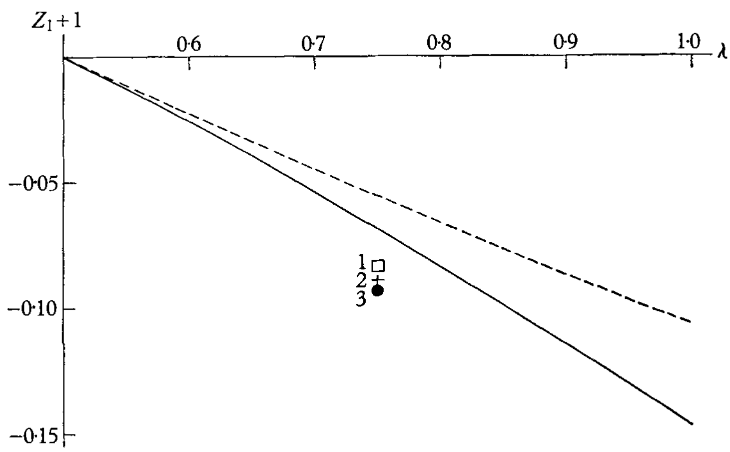

FIGURE 3. Minimum values of $Z_1 + 1$ (i.e. maximum depth reached by the discontinuity) as a function of the parameter λ: - - - - -, perturbation method; ———, method of Whitham and Rościszewski. Values measured in three experiments are plotted as numbered points.

The analysis is performed along the same lines as in the case of the zero-order perturbation coefficient, and expressions analogous to (39), (40) and (41) are obtained. A new estimate for the jump motion is found by combining the analogue of (41) (which relates W_0 and W_1 upon a descending characteristic) with (26), (28) and (29), and eliminating R_0, R_1 and W_1. Parametric relations are found for Z_1 and T_1 in terms of W_0. If $(2\lambda)^{\frac{1}{2}} = p$,

$$Z_1 + 1 = \frac{W_0}{p}\left\{(1-p) + (1+p)\frac{p^2}{4W_0^2}\left(\frac{2W_0-\lambda}{p-\lambda}\right)^2\right\}, \tag{47}$$

$$\frac{\lambda T_1}{1+p} = \left\{2 + \frac{1-p}{1+p}\left(\frac{p-\lambda}{p}\right)^2\right\}\log\left(\frac{p-\lambda}{2W_0-\lambda}\right) + \log\left(\frac{2W_0}{p}\right) + \tfrac{1}{2}(p-\lambda)\frac{1-p}{1+p}\left(\frac{p-\lambda}{2W_0-\lambda} - 1\right). \tag{48}$$

This approximate solution also has the property that the maximum depth of descent occurs at a finite value of T_1 when $\lambda > \frac{1}{2}$.

The estimates for the limiting values of Z_1 from the two methods are plotted in figure 3. Both estimates touch the straight line

$$Z_1 = -1 - \tfrac{1}{4}(\lambda - \tfrac{1}{2}). \tag{49}$$

Since the ultimate value of Z_1 is greater than unity when $\lambda > \frac{1}{2}$, it is clear that a certain amount of 'overshooting' must occur for that case. If $0 < \lambda < \frac{1}{2}$, the discontinuity would appear to 'undershoot'. However, the final state of the fluid would then involve a density gradient in excess of the critical value, for which the system would be unstable.

5. Experimental results

The motion of a discontinuity descending into still fluid of constant density is most readily studied by means of Taylor's experiment, using a vertical tube of circular cross-section filled with porous material. A description of the experimental procedure has been given by Wooding (1959).

Expt. no. ...	1	2	3
Solute	Potassium permanganate	Sodium sulphate, Meth. blue	Sodium sulphate, Meth. blue
Mean temp. (°C)	22·5	14·0	21·0
Tube rad. b (cm)	0·35	0·504	0·504
Porosity, ϵ	0·36	0·36	0·36
Permeability, $10^7 k$ (cm^2)	3·2	3·0	3·47
Viscosity, $10^2 \mu$ (poise)	1·00	1·20	1·02
Density diff.			
$\Delta\bar{\rho}$ (g/cm^3)	0·040	0·0192	0·0222
A (cm)	15·3	15·5	25·7
X_0 (cm)	−0·58	−0·59	−0·69
$10^4 B$ (sec^{-1})	−0·857	−0·316	−0·297
$t_0/3600$ (sec)	−0·1	−0·30	1·0
λ	0·75	0·75	0·74
$(Z_1)_{\text{min.}}$	−1·083	−1·090	−1·093

TABLE 1. Values of physical parameters obtained from three experiments on free convection in tubes filled with saturated porous material.

Three sets of experimental data have been obtained from experiments of this type, and are summarized in tables 1 and 2. Table 1 gives the measured values of certain relevant physical parameters for each experiment. In each case, the temperature did not vary more than about $\pm 2°$ C from the recorded mean value. The mean tube radius, b, was found by measuring the volume of water required to fill a given length of tube, while the porosity ϵ was obtained by the same technique when the tube contained Ballotini glass beads of about 0·2 mm diameter. The mean viscosity μ was measured from Poiseuille-type experiments, conducted at the same temperature in each case as the corresponding experiment with porous material. A density-bottle method was used to determine the initial difference in density, $\Delta\bar{\rho}$, between the aqueous solution in the upper reservoir and the water saturating the porous medium in the vertical tube.

Table 2 gives the distance, $-X_3$, of the leading edge of the discontinuity from the upper surface of the porous medium as a function of the time t since the start of each experiment.

Expt. 1		Expt. 2		Expt. 3	
Elapsed time (h)	Depth, $-X_3$ (cm)	Elapsed time (h)	Depth $-X_3$ (cm)	Elapsed time (h)	Depth, $-X_3$ (cm)
0	0	0	0	0	0
0·83	2·95	1·03	2·0	2·94	3·1
0·92	3·3	2·7	4·2	3·5	4·5
1·0	3·65	4·0	5·8	4·0	5·65
1·17	4·3	5·55	7·3	4·55	7·1
1·33	4·95	7·0	8·5	5·05	8·25
1·5	5·5	8·5	9·5	5·5	9·3
1·7	6·1	10·0	10·5	6·0	10·3
2·42	8·0	11·45	11·25	7·25	12·5
3·0	9·2	13·0	11·9	7·5	12·8
3·75	10·4	14·5	12·5	8·0	13·55
4·25	11·1	16·0	12·95	8·5	14·25
5·08	12·1	18·0	13·5	10·0	16·0
5·42	12·35	20·1	14·0	12·5	18·35
5·58	12·55	21·7	14·3	15·0	20·0
6·4	13·15	22·5	14·45	18·0	21·5
7·0	13·55	23·2	14·55	20·4	22·5
7·67	13·9	24·0	14·7	22·0	23·1
9·58	14·7	25·0	14·85	27·6	24·65
11·0	15·15	26·32	15·0	31·6	25·35
13·0	15·5	27·47	15·15	32·5	25·45
15·0	15·75	28·22	15·25	37·5	26·0
18·0	15·9	29·9	15·4	42·5	26·4
21·67	16·0	32·05	15·6	47·5	26·75
		33·65	15·8	52·5	27·0
		36·0	15·9	56·8	27·1
		44·5	16·2	62·5	27·25
		46·5	16·3	67·5	27·3
				73·7	27·4

TABLE 2. Time-depth measurements from three experiments on convection in vertical tubes filled with saturated porous material

To match the approximate solutions of § 4 to these results the relationships

$$\left.\begin{aligned} Z_1 &= (X_3+X_0)/A, \\ \tfrac{1}{2}\lambda T_1 &= -B(t-t_0) \end{aligned}\right\} \tag{50}$$

are assumed, where A, B, X_0 and t_0 are additional parameters. From the definitions of Z_1, T_1 (with $\beta = 0$) and the critical Rayleigh number

$$\alpha^2 = 3{\cdot}39 = \gamma g k b^2/\kappa\mu,$$

it is found that

$$\left.\begin{aligned} A &= \Delta\bar{\rho}/\gamma, \\ B &= -\tfrac{1}{2}\gamma\lambda g k/\epsilon\mu, \end{aligned}\right\} \tag{51}$$

where γ denotes the density gradient of the fluid at neutral stability. When X_0 is non-zero, the origin $Z_1 = 0$ corresponds to a 'virtual origin' which is slightly displaced from the top of the porous medium. The existence of this parameter is not surprising since the theoretical treatment of the end conditions is very approximate. The value of t_0 is adjusted to compensate for variations in the rate of development of the initial finite disturbance.

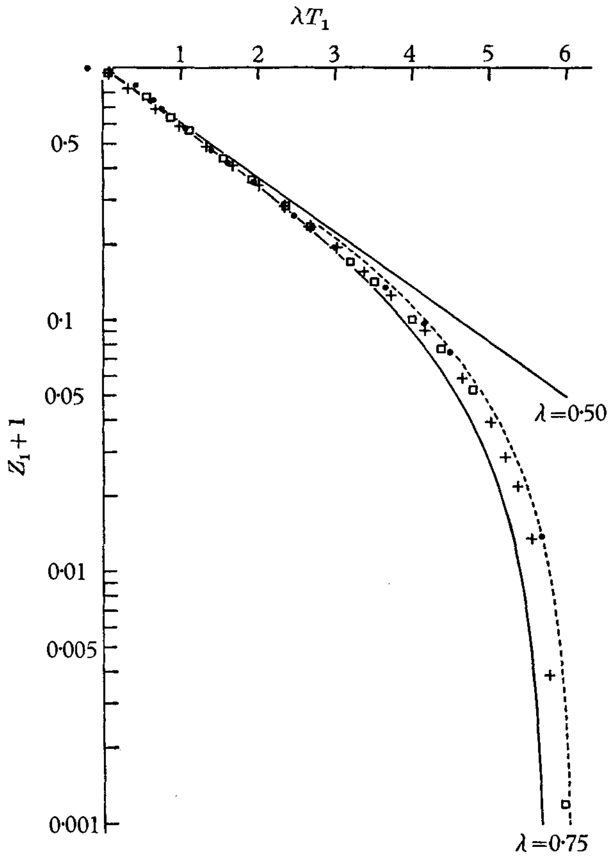

FIGURE 4. Experimental values of Z_1+1 compared with approximate theoretical values as functions of λT_1 for $\lambda = 0{\cdot}5$ and $0{\cdot}75$: ------, perturbation method; ———, method of Whitham and Rościszewski; □, Expt. 1; +, Expt. 2; ●, Expt. 3.

Approximate values for the above parameters have been determined graphically, using the simple expression (46) to equate the velocities in the early stages of the motion, and are listed in table 1 for each of the three experiments. The value of λ is calculated from (51) for each case.

The relations (50), with the values of the parameters given in table 1, have been used to convert the experimental points to the co-ordinate system $(Z_1+1, \lambda T_1)$. These points are plotted in figure 4. Curves derived from the approximate theoretical results of § 4 for the jump motion are drawn for the cases $\lambda = 0{\cdot}5$ and $0{\cdot}75$; the latter is in good agreement with the experimental data.

Values of the maximum depth of descent measured experimentally are given in table 1 in dimensionless form ($(Z_1)_{\text{min.}}$), and are plotted with abscissa $\lambda = 0{\cdot}75$ in figure 3. The observed values exceed the predicted theoretical value by 2 or 3 %.

6. Conclusions

The simple theory described in this paper has been found to give results which are in satisfactory agreement with experiment. However, since no fewer than five experimental parameters were determined from the experimental data, some shortcomings of the theory may have been concealed. These expected difficulties are (i) the fact that the Péclet number exceeded $O(1)$ during the early stages of the motion in the experiments, (ii) the neglect of changes in viscosity and diffusivity with changes in solute concentration, (iii) the neglect of longitudinal diffusion, which may have become significant at low convection velocities, and (iv) shortcomings in the approximate methods of calculating the jump motion.

In any case, it appears that the descending fluid can 'overshoot' the point of neutral stability by something less than 10 %. This factor should therefore be considered if convection methods are employed to obtain physical measurements in stability problems.

The author wishes to acknowledge with gratitude several valuable discussions with Sir Geoffrey Taylor, F.R.S., Dr P. G. Saffman, Dr J. T. Stuart, Dr B. R. Morton and Mr C. H. Bosanquet. Some calculations were performed using the IBM computer at the Treasury, Wellington, with the assistance of Mr E. W. Jones. Part of this paper has been extracted from a doctoral dissertation written at Cambridge University under the supervision of Dr P. G. Saffman whilst the author was on leave from the New Zealand Department of Scientific and Industrial Research.

REFERENCES

Courant, R. & Friedrichs, K. O. 1948 *Supersonic Flow and Shock Waves.* New York: Interscience.

Lighthill, M. J. & Whitham, G. B. 1955 *Proc. Roy. Soc.* A, **229**, 281.

Rościszewski, J. 1960 *J. Fluid Mech.* **8**, 337.

Stoker, J. J. 1948 *Comm. Appl. Math.* **1**, 1.

Taylor, Sir Geoffrey 1954 *Proc. Phys. Soc.* B, **67**, 857.

Whitham, G. B. 1958 *J. Fluid Mech.* **4**, 337.

Wooding, R. A. 1959 *Proc. Roy. Soc.* A, **252**, 120.

Wooding, R. A. 1960 *J. Fluid Mech.* **7**, 501.